Signal Analysis & Communication ECE355

Ch. 5.1: Discrete Time Fourier Transform (DTFT)

Lecture 24

02-11-2023

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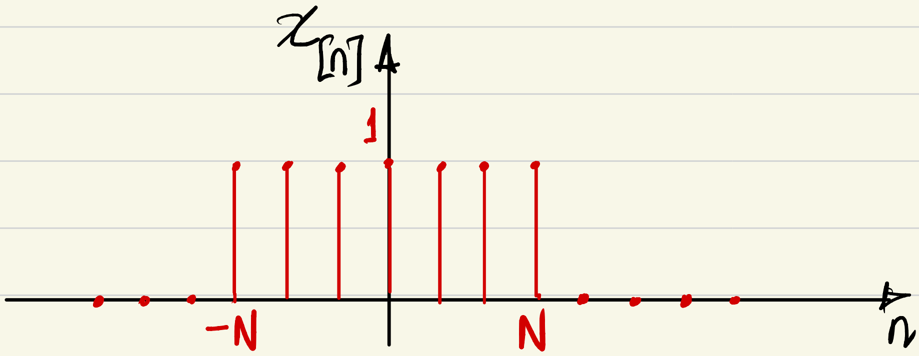
Example 2

$$x_{[n]} = \begin{cases} 1 & , \ |n| \leq N \\ 0 & , \ |n| > N \end{cases}$$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{[n]} e^{-j\omega n}$$

$$= \sum_{n=-N}^{N} e^{-j\omega n}$$

$$\overset{m=n+N}{=} \left[\sum_{m=0}^{2N} e^{-j\omega m} \right] \cdot e^{j\omega N}$$

$$= \left[\frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}} \right] \cdot e^{j\omega N}$$

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

$$= \frac{e^{j\omega N} - e^{-j\omega(N+1)}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega/2}\left(e^{j\omega(N+1/2)} - e^{-j\omega(N+1/2)}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \frac{e^{-j\omega/2}\left(e^{j\omega(N+1/2)} - e^{-j\omega(N+1/2)}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \frac{\sin(\omega(N+1/2))}{\sin(\omega/2)} \quad \text{— (1)}$$

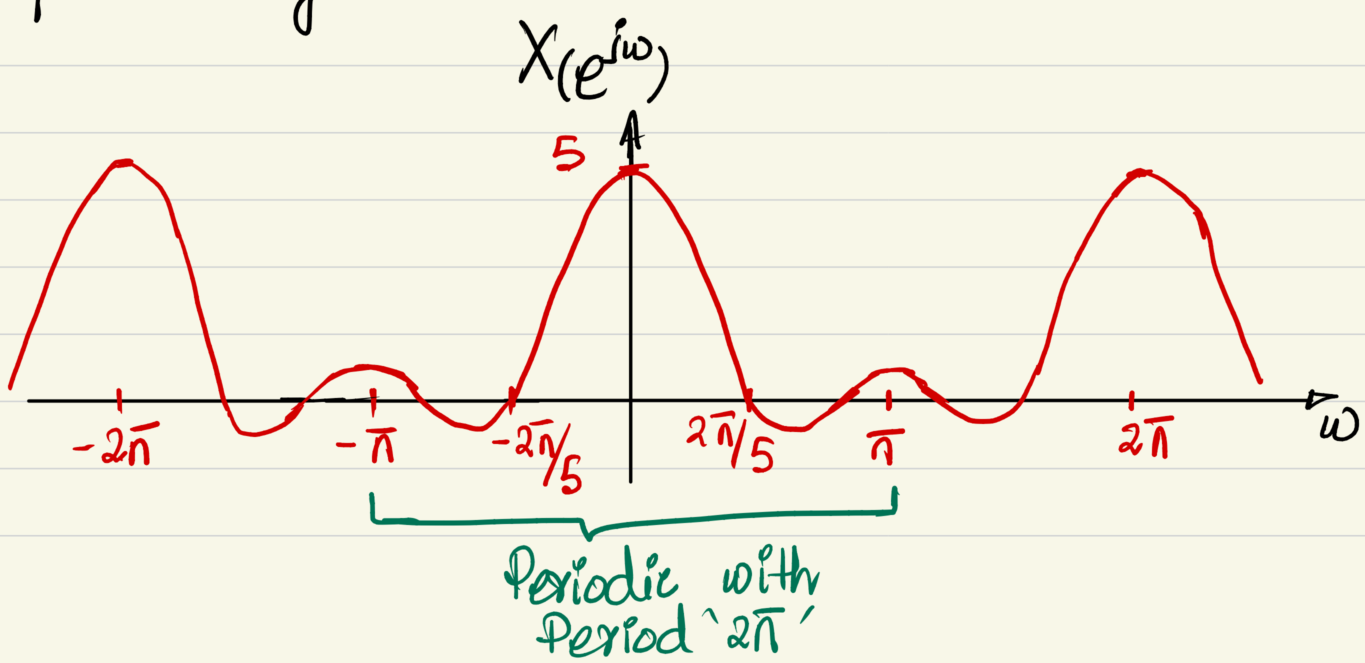
$$= \frac{\sin(\omega(5/2))}{\sin(\omega/2)} \qquad \text{with } N=2$$

$$= \frac{\sin\left(\omega\left(\frac{5}{2}\right)\frac{\pi}{\pi}\right)}{\sin(\omega/2)\,\frac{\omega 5}{2\pi}\cdot\pi} \times \omega\frac{5}{2} \qquad \text{sinc } f^n$$

$$= \frac{5}{2}\,\frac{\omega}{\sin\left(\frac{\omega}{2}\right)}\ \text{sinc}\ \frac{5}{2\pi}\omega \quad \text{— (2)}$$

at $\omega=0$, $= 2$ (l'Hopital rule)

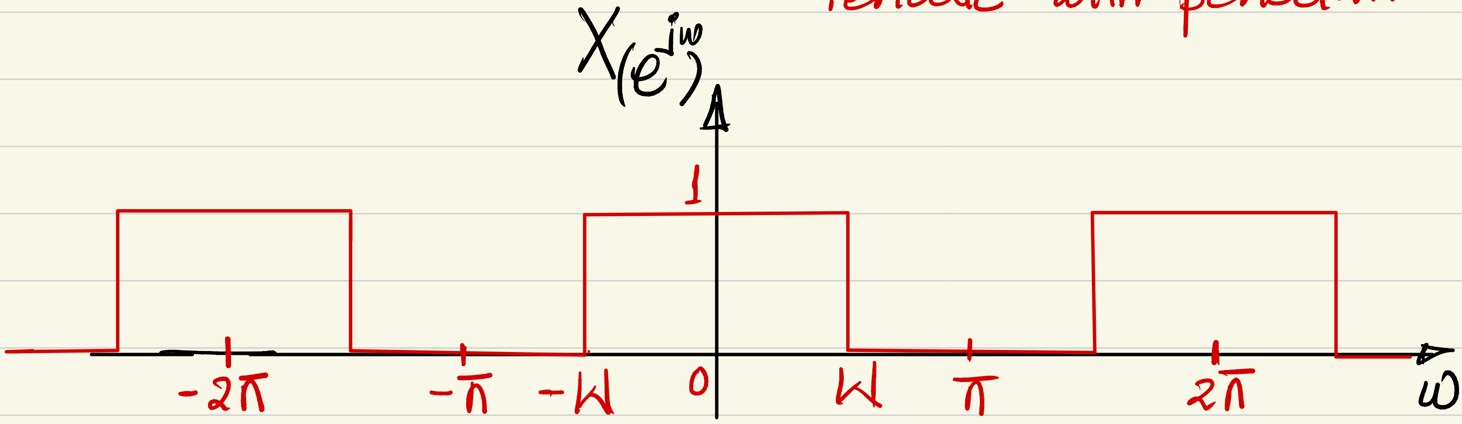
- Eqn (2) $\Rightarrow$ sinc function, the difference is it's periodicity of 2π.



Example 3

$$X(e^{j\omega}) = \begin{cases} 1 , & |\omega| \leq W \leq \pi \\ 0 , & W < |\omega| \leq \pi \end{cases}$$

Periodic with period 2π

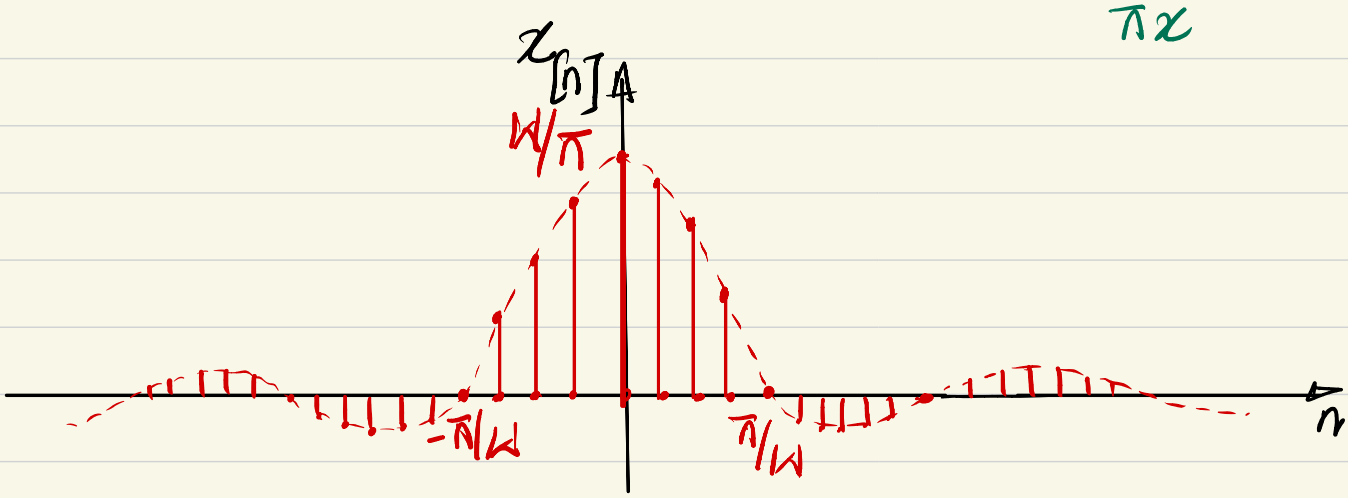


$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega n} d\omega$$

$$= \frac{\sin(Wn)}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$$

Compare it with $\frac{\sin \pi x}{\pi x} = \operatorname{sinc}(x)$



Example 2 & 3 $\Rightarrow$ $\boxed{\text{Pulse} \longleftrightarrow \text{Sinc Function}}$

Example 4

a) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n}$$

$$= e^{j0} \sum_{n=-\infty}^{\infty} \delta[n]$$

$$= 1.$$

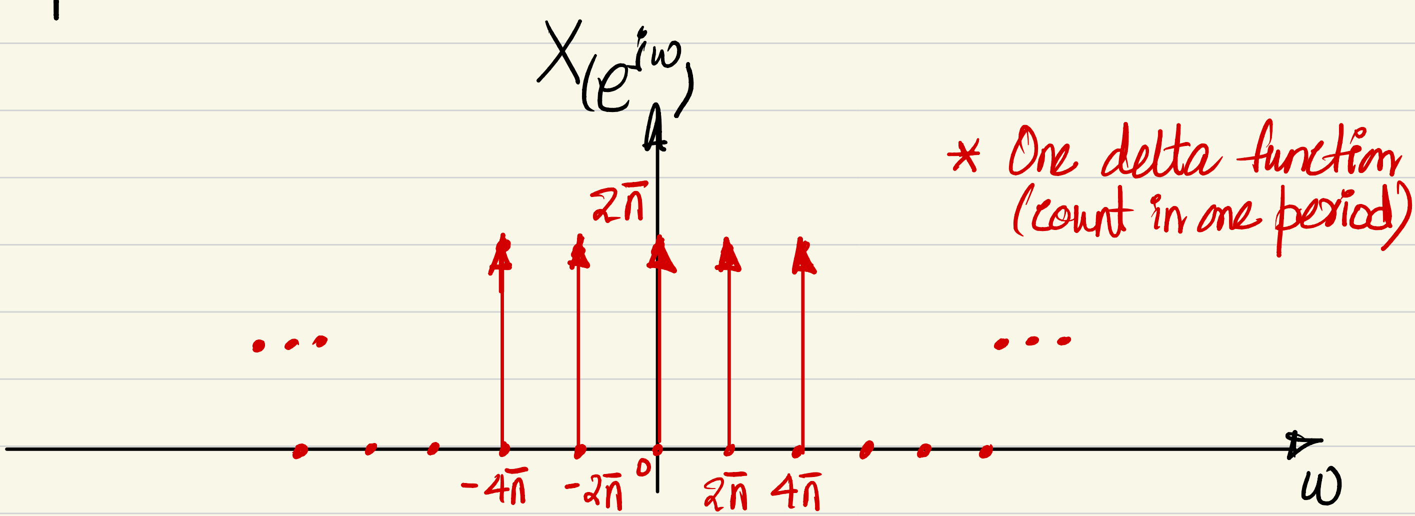
b) $x[n] = \delta[n-k]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-k] e^{-j\omega n}$$

$$= e^{-j\omega k} \sum_{n=-\infty}^{\infty} \delta[n-k]$$

$$= e^{-j\omega k}$$

Example 5



$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi \, \delta(\omega - 2\pi l)$$

$$x_{[n]} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{(e^{j\omega})} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \, \delta_{(\omega)} e^{j\omega n} d\omega$$

one delta f^n in one period

$$= 1, \quad \forall n$$

Example 4 (a) & 5 $\Rightarrow$ $\boxed{\text{Impulse} \longleftrightarrow \text{Constant DC}}$

Ch. 5.2: DTFT FOR PERIODIC SIGNALS

- As in the CT case, DT Periodic signals can be incorporated within the framework of the DTFT by interpreting the transform of a periodic sig. as weighted (a_k) impulse in freq.-domain.

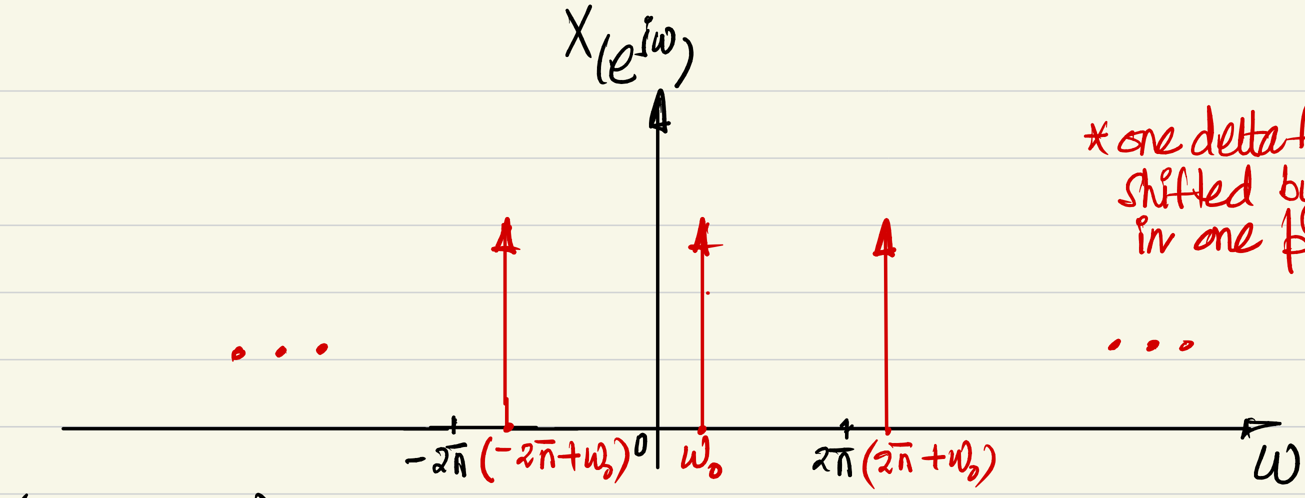
- Recall, in CT:

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi \, \delta(\omega - \omega_0)$$

Freq. shift.

- In DT, however, the transform must be periodic in 'ω' with period 2π.
- This suggest that $\mathcal{F}\{e^{j\omega_0 n}\}$ should have impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$ & so on.
- Therefore,

$$\mathcal{F}\{e^{j\omega_0 n}\} = X_{(e^{j\omega})} = \sum_{l=-\infty}^{\infty} 2\pi \, \underbrace{\delta_{(\omega - \omega_0 - 2\pi l)}}_{\text{Periodic with period } 2\pi}$$



* one delta function shifted by 'ω_0' in one period.

(Conversely)

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 2\pi \delta(\omega - \omega_0) e^{j\omega n} d\omega$$

$$= e^{j\omega_0 n}.$$

- Now consider a periodic sig. $x[n]$ with period 'N' & with FS representation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

- It's FT is given as

$$X(e^{j\omega}) = \sum_{k=\langle N \rangle} a_k \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - k\omega_0 - 2\pi l)$$

$\omega_0 = 2\pi/N$

$$= \sum_{k=\langle N \rangle} a_k \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - k\omega_0 - N\omega_0 l)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \underbrace{a_{k+lN}}_{\text{periodic with 'N'}} 2\pi \delta(\omega - \underline{(k+lN)\omega_0})$$

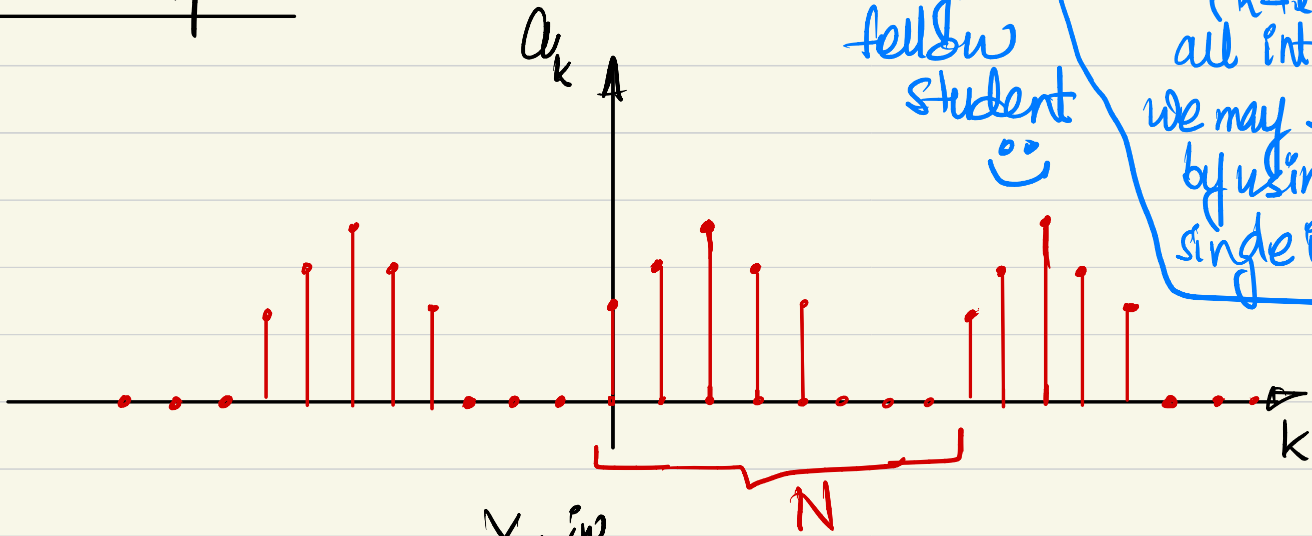
$$= \sum_{k=-\infty}^{\infty} \underbrace{2\pi a_k}_{\text{Same as that of CTFT case.}} \delta(\omega - k\omega_0)$$

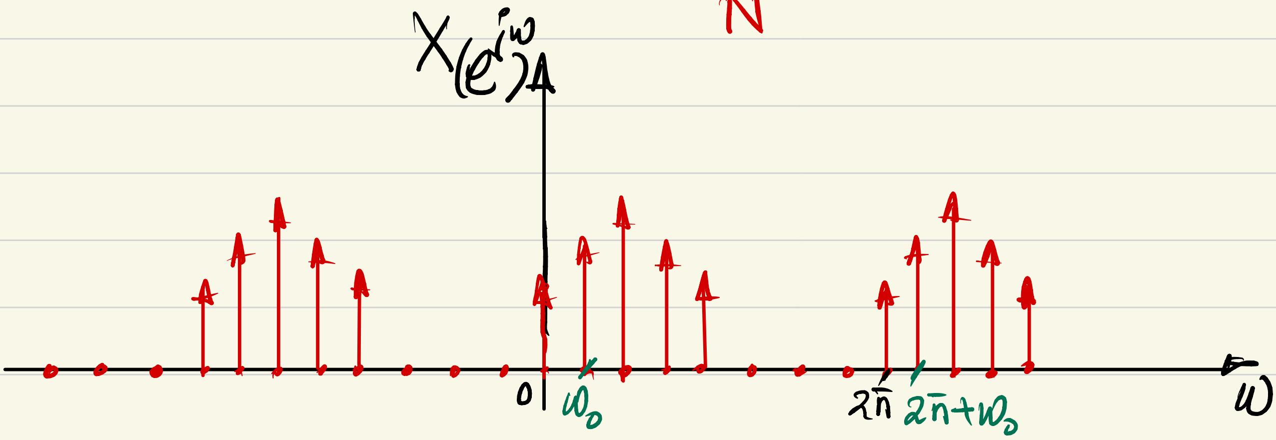
Either sum period by period OR sum altogether

* by our fellow student ☺

OR $(k+lN)$ all integers we may simplify by using a single integer

For example

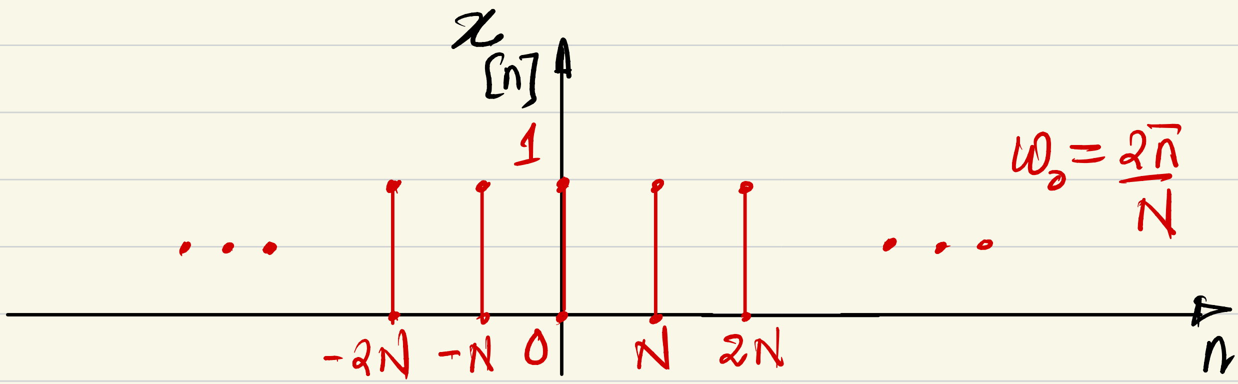




Example

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

DT Periodic



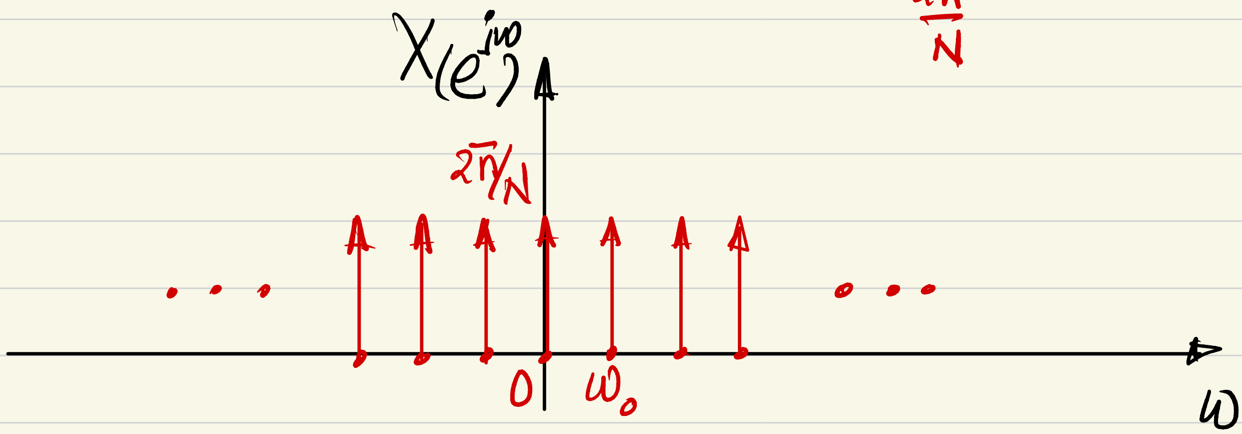
$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \delta[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}, \quad \forall N$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta(\omega - k\underbrace{\omega_0}_{\frac{2\pi}{N}})$$

Impulses scaled by $a_k \cdot 2\pi$



"PICKET FENCE MIRACLE"

★ Impulse train in time-domain is Impulse train in freq.-domain.